SIMILARITY OF NON-NEWTONIAN FLOWS. VI.*

A PSEUDOSIMILARITY APPROACH TO THE FLOW OF THE GENERALIZED NEWTONIAN FLUID IN THE ENTRANCE REGION OF A TUBE

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A generalization is given of the Schiller's approach to the flow in the entrance region of a tube at high Reynolds numbers of an arbitrary model of the generalized Newtonian fluid. Numerical solutions are presented for the Eyring model. The generalized Reynolds number of the flow in the tube and the apparent flow index introduced in a standard manner can be utilized for a generalized solution of the problem of the entrance region by a pseudosimilarity approach. The plausibility of this method is confirmed by comparing the results for the power-law, the Eyring and the Bingham models.

The theory of the non-Newtonian flow in the entrance region of a tube has been represented predominantly by papers using the power-law model of the generalized Newtonian fluid[1-5] (GNF) to describe its rheological properties. The paper of Bogue[1] as well as a similar analysis of Tyabin and Centkovskij[2] are based on the Kármán–Pohlhausen method of the integral balances. Tomita[3] has used the variation calculus. A more recent paper by Collins and and Schowalter[4] applies the Schlichting's method *i.e.* a combined solution by linear perturbation and the boundary layer method. A recent paper of Christiansen and Carter[5] attempts solution to the complete set of the flow equations by a finite difference method. The last two papers begin with relatively realistic mathematical models of the hydrodynamics in the entrance region and their results are comparable with the exact solutions for the Newtonian flow[6,7].

For more sophisticated rheological models, such as *e.g.* the Bingham[8] and the Eyring[9] model of the GNF and a simple model of viscoelasticity of the Rivlin–Ericksen type[10], only various solutions based on the Kármán–Pohlhausen method of the integral balances have been made available. This method was used for the first time by Schiller[11] for a Newtonian fluid.

In the presented paper the Schiller's hydrodynamic model[11] serves to examine to what extent the results obtained for the power-law model of the GNF (for which numerous exact solutions to the problem of the entrance-flow region are available) may be used in case that the viscosity function cannot be characterized by a power expression. For this purpose, a method termed the pseudosimilarity approach

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and described in earlier communications[12–14] will be used. This method compares the results of a given hydrodynamic problem for various rheological models in terms of an apparent flow index, n^*, and a generalized Reynolds number, Re^*. The manner in which these quantities are introduced is independent of the type of the viscosity function and starts from the relation between macroscopic characteristics of a given flow pattern[13,14], which are easy to obtain by experiment.

A basis of the pseudosimilarity approach applied to the GNF is the assumption that the flow of two different GNF possessing possibly substantially different viscosity functions will exhibit approximately identical pattern of the normalized velocity, the isotropic pressure and the shear stress tensor fields provided that the corresponding values of n^* and Re^* in both cases are equal.

This assumption is tested in this paper on the computed results of the hydrodynamics of the entrance-flow region in a tube for the Bingham, the Eyring and the power-law models of the GNF.

A Generalization of the Schiller's Approach to the GNF

To solve the problem of the hydrodynamics of a Newtonian flow in the entrance region of the tube, Schiller has introduced a number of simplifying assumptions formulated so as to permit the extension of this approach to an arbitrary model of the GNF[15]:

1) The piston flow at the inlet

$$v_z = U \quad \text{for} \quad z = 0\,, \tag{1}$$

transforms into a fully developed laminar profile, $v_D(r)$, within a finite section – the entrance region – of the tube z_E long,

$$v_z = v_D(r) \quad \text{for} \quad z \geqq z_E\,. \tag{2}$$

2) The entrance-flow region may be divided into a region of the boundary layer near the wall, and the core region surrounding the axis of the tube. In the core region one may assume that all deformation stresses are negligible in comparison with the effect of inertia forces. Accordingly, the hydrodynamics of the core flow may be stated in terms of the Euler equations of the flow. With respect to the boundary condition (*1*) and this last assumption, the flow within the core is a potential one and its velocity field is given by

$$v_z = v_0(z) \quad \text{for} \quad r < R - \delta(z)\,, \tag{3}$$

where $\delta(z)$ is the thickness of the boundary layer. The pressure drop within the core is given by the Bernoulli equation

$$\mathrm{d}P/\mathrm{d}z = (\varrho/2)\,\mathrm{d}v_0^2(z)/\mathrm{d}z\,. \tag{4}$$

3) Besides the inertia forces and the effects of the isotropic pressure, the only important component of the shear stress tensor in the boundary layer, *i.e.* for $R - \delta(z) < r < R$ is

$$\tau_{rz} = \tau[-\partial v_z/\partial r]\,, \tag{5}$$

as follows from the Prandtl equations of the boundary layer, which may be written in the form

$$\varrho\left(v_z \frac{\partial v_z}{\partial z} + v_r \frac{\partial v_z}{\partial r}\right) = -\frac{\partial P}{\partial z} + \frac{1}{r}\frac{\partial(r\tau_{rz})}{\partial r}\,, \quad 0 = \frac{\partial P}{\partial r}\,. \tag{6a,b}$$

4) Since we content ourselves with determining the macroscopic characteristics and do not seek a detailed description of the velocity field, the problem is thus reduced to finding solution to the boundary layer equation which may be solved by the approximate Kármán–Pohlhausen method. This method consists in that we do not require that the velocity field satisfy the momentum balance in each point (*6a*), but rather its integral form averaged over the tube cross-section:

$$\varrho \frac{\mathrm{d}}{\mathrm{d}z}\int_0^R v_z^2 r\,\mathrm{d}r = -\frac{\mathrm{d}P}{\mathrm{d}z}\frac{R^2}{2} + R\tau\left[\left.\frac{\partial v_z}{\partial r}\right|_{r=R}\right]. \tag{7}$$

The approximate solution, of course, has also to satisfy the continuity equation

$$UR^2/2 = \int_0^R v_z r\,\mathrm{d}r \tag{8}$$

and the boundary conditions, *i.e.* the no-slip condition: $v_z = 0$ for $r = R$, the axial symmetry condition: $\mathrm{d}v_z/\mathrm{d}r = 0$ for $r = 0$, and the conditions (*1*) and (*2*).

There is an infinite number of alternatives how to superimpose a velocity field satisfying the above listed assumptions; the most common way is to assume the similarity of the velocity profiles within the boundary layer:

$$v_z(z, r) = v_0(z)\,f(y)\,, \tag{9}$$

where y is a normalized radial coordinate in the boundary layer

$$y = \frac{r - R + \delta(z)}{\delta(z)} \tag{10}$$

and $f(y)$ is a normalized velocity profile in the boundary layer which from the kinematic boundary conditions has to satisfy the following constraints: $f(0) = 1$, $f(1) = 0$, $f'(0) = 0$, $f''(y) < 0$. It is customary to choose polynomials for $f(y)$, but it seems natural to choose $f(y)$ such that it transforms continuously in $z = z_E$ into the developed velocity profile given by Eq. (*2*), *e.g.*

$$f(y) = v_D(yR)/U_{max} \tag{11}$$

since for $z = z_E$ it is $\delta = R$ and from Eq. (*10*) thus $y = r/R$.

Two parameters remain undetermined in the superimposed expressions for the velocity field (*9*)–(*11*): The maximum velocity $v_0(z)$ and the thickness of the boundary layer $\delta(z)$. These parametric functions can be chosen so as to satisfy the integral balances given in Eqs (*7*) and (*8*). Substituting for v_z from Eq. (*9*) into the continuity equation (*8*) and after some manipulation one obtains an algebraic relation between v_0 and δ. Eliminating the pressure gradient $(-dP/dz)$ by substituting from Eq. (*4*) into (*8*) and integrating with the velocity profile given by Eqs (*9*)–(*11*) one obtains a differential equation in v_0, δ and z. These equations may be recast in a more convenient form by introducing a normalized thickness of the boundary layer

$$\delta^* = \delta/R \tag{12}$$

namely:

$$U/v_0 = 1 - 2\alpha_1\delta^* + \alpha_2(\delta^*)^2 , \tag{13}$$

$$\frac{d/d\delta^*\{[1/4 + \beta_1\delta^* + \beta_2(\delta^*)^2]/[1 - 2\alpha_1\delta^* + \alpha_2(\delta^*)^2]^2}{\tau\{[-f'(1)\, U/R]/[\delta^*(1 - 2\alpha_1\delta^* + \alpha_2(\delta^*)^2} \, d\delta^* = \frac{1}{\varrho U^2}\frac{dz}{R} , \tag{14}$$

where the constants

$$\alpha_1 = 1 - \int_0^1 f(y)\, dy , \quad \alpha_2 = 1 - 2\int_0^1 f(y)(1 - y)\, dy , \qquad (15), (16)$$

$$\beta_1 = -1 + \int_0^1 f^2(y)\, dy , \quad \beta_2 = 1/2 - \int_0^1 f^2(y)(1 - y)\, dy \qquad (17), (18)$$

depend only on the function $f(y)$, *i.e.* on the type of the fully developed velocity profile $v_D(r)$.

Eq. (*14*) is integrated with the aid of the initial condition $z = 0$ for $\delta^* = 0$. The value corresponding to $\delta^* = 1$ gives the length of the entrance region $z = z_E$. The pressure drop within the entrance region according to Eq. (*4*) can be calculated from:

$$P(0) - P(z_E) = \varrho \frac{U_{max}^2 - U^2}{2} . \tag{19}$$

The Hagenbach's correction factor, C, gives the excess pressure drop due to the entrance-flow effect in this region over the pressure drop of a fully developed flow for which we have

$$-\left.\frac{\mathrm{d}P}{\mathrm{d}z}\right|_{z\to\infty} = \frac{2}{R}\tau_W\,. \qquad (20)$$

The definition of the factor C may be written for instance in the form:

$$C = \left[P(0) - P(z_E) - z_E\tau_w 2/R\right] 2/(\varrho U^2) = (\Delta P)_E\, 2/(\varrho U^2)\,, \qquad (21)$$

where $(\Delta P)_E$ stands for the excess pressure drop, *i.e.* the difference between the total pressure drop in the entrance region and that of the region with fully developed flow of equivalent length. Thus, using Eq. (*19*):

$$C = (U_{max}/U)^2 - 1 - 16(z_E/R)\,\tau_w/(4\varrho U^2)\,. \qquad (22a)$$

The Dimensionless Formulation

The Schiller's approach enables the problem to be solved in an approximative manner in terms of known characteristics of the fully developed flow in the tube. These are fully determined by the viscosity function of the liquid $D[\tau]$, or $\tau[D]$, its density, the radius of the tube, R, and by the average velocity of the flow, U, or the pressure gradient from Eq. (*20*).

In case of the laminar flow the relations between τ_w, U and R are generally known expressions

$$\tau_w = \mathfrak{P}[\mathfrak{F}]\,, \quad \text{or} \quad \mathfrak{F}[\mathfrak{P}] = \frac{4}{\mathfrak{P}^3}\int_0^{\mathfrak{P}} D[\tau]\,\tau^2\,\mathrm{d}\tau\,, \qquad (23), (24)$$

where

$$\mathfrak{F} = \frac{4U}{R} = \frac{4Q}{\pi R^3}\,. \qquad (25)$$

An arbitrary viscosity function can be normalized[16] by introducing at most two dimensional parameters D_I, τ_I possessing character of material constants of a given liquid. The normalized viscosity function may then be written *e.g.* in the form

$$\vartheta[p] = \tau[pD_I]/\tau_I \quad \text{or} \quad p[\vartheta] = D[\vartheta\tau_I]/D_I\,. \qquad (26a), (26b)$$

Under the laminar flow when $\tau_w = \mathfrak{P}$ the relation between $\mathfrak{F}$ and τ_w is only an integral transformation of the viscosity function according to Eq. (*24*) and it seems therefore natural to normalize it in accord with Eq. (*26b*) by introducing the following dimensionless parameters A and B.

$$A = \tau_w/\tau_I\,, \quad B = \mathfrak{F}/D_I = (4U/R)/D_I\,. \qquad (27a), (27b)$$

Between A and B under the laminar flow we have from Eq. (*24*) the following relation

$$B[A] = 4\int_0^1 p[Ay]\, y^2\, dy\,. \qquad (28)$$

In the same way one can write corresponding expressions for the normalized velocity field in the boundary layer by means of the given viscosity function $p[\vartheta]$ *e.g.*

$$f(y) = \int_y^1 p[Ay]\, dy \Big/ \int_0^1 p[Ay]\, dy\,, \qquad (29)$$

$$1/\gamma_1 = \int_0^1 f(y)\, 2y\, dy = U/U_{max} = (B/4) \Big/ \int_0^1 p[Ay]\, dy\,, \qquad (30)$$

$$-f'(1) = 4\gamma_1 p[A]/B\,. \qquad (31)$$

On introducing a new dimensionless axial variable as

$$\xi = \frac{z}{R}\frac{\mathfrak{P}}{4\varrho U^2} \qquad (32)$$

the basic equation (*14*) to the Schiller's analysis of the entrance flow region may be recast in dimensionless form as

$$\frac{d/d\delta^*\{(1/4 + \beta_1\delta^* + \beta_2(\delta^*)^2)/(1-2\alpha_1\delta^* + \alpha_2(\delta^*)^2)^2}{\vartheta[(p[A]/\gamma_1)/\delta^*(1-2\alpha_1\delta^* + \alpha_2(\delta^*)^2)]/A}\, d\delta^* = d\xi \qquad (33)$$

Similarly the Hagenbach's correction factor may be given as

$$C = \gamma_1^2 - 1 - 16\xi_E\,, \qquad (22b)$$

where $\xi = \xi_E$ is found by integration according to Eq. (*33*) in the limits $\delta^* = 0$, $\delta^* = 1$.

For a given viscosity function $\vartheta[p]$ the dimensionless statement of the problem thus contains a single independently variable parameter for which one may choose either the dimensionless pressure drop A or the dimensionless volume flow rate B.

An exception in this respect is the power-law model

$$\vartheta = p^{n} \tag{34}$$

because of its automorphous properties[12]. In this case the denominator of Eq. (33) can be simplified to give an expression

$$\vartheta[\cdots]/A = \left(\frac{\gamma_1}{\delta^*(1-2\alpha_1\delta^* + \alpha_2\delta^{*2})}\right)^{n}, \tag{35}$$

which does not contain A any more. The developed velocity profile contains also a single parameter: the flow index (the simplex of rheological similarity[16])

$$f(y) = 1 - y^{(1+1/n)}. \tag{36}$$

Consequently, the constants $\alpha_1, \beta_2, \beta_1, \beta_2$ are functions of n only. Likewise, the resulting ξ_E and C are functions of n only.

The Pseudosimilarity Approach

The analysis of the hydrodynamics of non-Newtonian liquids by pseudosimilarity approach[14] may be always divided into two steps: The first step is the solution of the problem for a suitable automorphous model[12] of given rheological properties; in case of the GNF thus for the power-law model. The second step is the search for an adequate interpretation of more realistic models of non-Newtonian behaviour in terms of parameters of the corresponding automorphous model and macroscopic parameters of the problem in question; in case of the GNF thus through a suitably defined apparent flow index, n^*, and an apparent coefficient of consistency, K^*.

The Schiller's solution to the problem of the entrance-flow region reduces to determining the boundary layer thickness as a function of the axial coordinate; the other characteristics being deduced directly from the fully developed laminar velocity profile. Accordingly, having introduced the pseudoparameters of the power-law model we still start from the characteristics of the fully developed laminar flow which we introduce by the relations

$$n^* \equiv \frac{\mathrm{d}\ln \mathfrak{P}}{\mathrm{d}\ln \mathfrak{F}} = \frac{\mathrm{d}\ln A}{\mathrm{d}\ln B} = \frac{1}{p[A]/B - 3}, \tag{37}$$

$$K^* \equiv \mathfrak{P}\left(\frac{3n^* + 1}{4n^*}\frac{4U}{R}\right)^{-n^*} \tag{38}$$

proposed by Metzner and Reed[17]. These relations have proved useful for correlating the experimental results of the turbulent flow of non-Newtonian liquids in a tube using the generalized Reynolds number Re*. This group has been defined earlier[17] as

$$\mathrm{Re}^* = 4U^2/K^* \left(\frac{3n^* + 1}{4n^*}\frac{4U}{R}\right)^{n^*}. \tag{39}$$

The parameter K^* does not appear in the normalized model of the entrance-flow region explicitly, but it is contained in the definition of the dimensionless axial coordinate, Eq. (*32*), which may be alternatively written as

$$\xi = \frac{z}{R}\frac{1}{Re^*}, \quad \text{where} \quad \mathrm{Re}^* = 4\varrho U^2/\mathfrak{P}[4U/R] \qquad (40), (41)$$

is equivalent to the generalized Reynolds number defined in Eq. (*39*).

Since the simplifying assumptions necessitated by the Schiller's approach provide for the inertia effect (represented by Re*) only by the choice of the dimensionless axial coordinate ξ, the assumptions of the kinematic pseudosimilarity reduce to the following relations

$$f(y) \approx f_a(y, n^*), \quad \delta^*(\xi) \approx \delta^*_a(\xi, n^*), \qquad (42), (43)$$

both being valid for an arbitrary $\vartheta[p]$. The assumptions of the dynamic pseudosimilarity according to Eqs (*4*), (*5*), (*13*) follow from Eqs (*42*) and (*43*).

The Hagenbachs's correction factor is justly regarded as the key to each entrance-flow problem and it is therefore important to check its presumed dynamic pseudosimilarity in the form

$$C \approx C_a(n^*), \quad \text{for an arbitrary} \quad \vartheta[p] \tag{44}$$

in the first place.

The Test of Pseudosimilarity

From the above it follows that although pseudosimilarity offers a generalized approach to non-Newtonian hydrodynamics, its character is one of a hypothesis, the validity of which has to be confirmed in each case by comparing the solution (mathematical solution or correlation of experimental data) for the power-law model with that for other models of the viscosity function. The reasons which make the Eyring model

$$\tau = \tau_1 \operatorname{arsinh}(D/D_1) \quad \text{or} \quad \vartheta = \operatorname{arsinh}(p) \tag{45a}$$

and the Bingham model

$$\tau = \tau_0 + \mu_B D \quad \text{or} \quad \vartheta = 1 + p \tag{45b}$$

best suited for the similarity test in the given sense (mainly for conspicuous differences between these two models and between each of them on one hand and the power-law model for an arbitrary n on the other hand) have been given in the preceding communications[13,14].

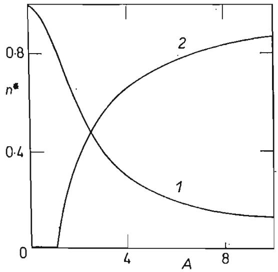

FIG. 1

Function $n^*(A)$ for the Eyring *1* and the Bingham *2* Model

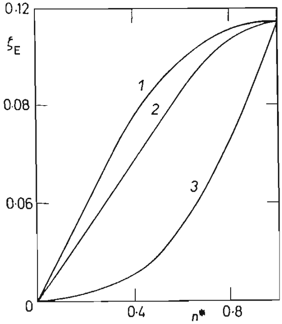

FIG. 2

The Length of the Entrance Region as a Function of the Apparent Flow Index

1 The Eyring model, *2* the power-law model, *3* the Bingham model.

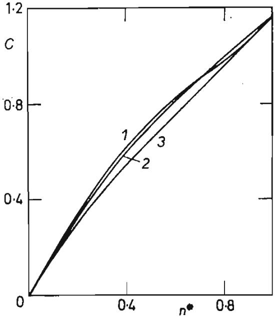

FIG. 3

The Hagenbach's Correction Factor as a Function of the Apparent Flow Index

1 The Eyring model; *2* the power-law model, *3* the Bingham model.

The solution of the entrance-flow region for the Bingham model by the Schiller's method has been published recently[8] and a similar solution for the Eyring model has been presented in an unpublished thesis[9]. The results of the latter are given in the appendix. Both cases are those of a non-automorphous model and accordingly, A (see Eq. (*27a*)), was introduced as a natural parameter. The relation between n^* and A, given by Eq. (*37*), is for both models shown graphically in Fig. 1. The Schiller's solution represented for a given viscosity function generally by the relations

$$f = f(y, A)\,, \quad \delta^* = \delta^*(\zeta, A)\,, \quad C = C(A) \tag{46a,b,c}$$

may be expressed for each of the models in the form where the independent parameter A is replaced by $n^*[A]$, see Fig. 1. This form enables a comparison to be made of the solutions for various models of the viscosity function having identical values of n^*, *i.e.* to test pseudosimilarity.

The pseudosimilarity of the fully developed velocity profiles in a tube has been studied *e.g.* in cit.[13] showing plots of the function $\gamma_1 f(y)$ and values of γ_1 for severa lvalues of n^*. Fig. 2 and 3 plot $\zeta_E(n^*)$ and $C(n^*)$ respectively in the range of pseudoplastic behaviour, *i.e.* within $0 \leqq n^* \leqq 1$.

DISCUSSION AND CONCLUSIONS

As it is known[13] the agreement of $f(y, n^*)$ for both models of the GNF in comparison with the power-law model is better than 5%. From Fig. 2 it is seen that $\xi_E(n^*)$ agrees quite well with the power-law model in case of the Eyring model, while for the Bingham model the normalized lengths of the entrance-flow region at equal n^* are substantially shorter. The kinematic pseudosimilarity in the entrance region may therefore be expected to exist for liquids with concave[13] viscosity functions, while the estimate of the length of the entrance region for viscoplastic liquids with convex viscosity functions is much too conservative.

A very good agreement of $C(n^*)$ for both models tested against $C_a(n)$ for the power-law model confirms the dynamic pseudosimilarity and substantiates conclusions that even results of more realistic hydrodynamic models of the entrance-flow for a power-law liquid may be used to make a pseudosimilarity estimate of the excess pressure drop for liquids with an arbitrary type of viscosity function.

The value of the results obtained by the Schiller's approach, as well as those obtained by other methods based on the equations of motion simplified in accord with the theory of the boundary layer, is somewhat lessened by the fact that these equations are adequate only at high values of Re*. At low Reynolds numbers become important also the radial components in the momentum balance in addition to some other axial terms. This fact manifests itself in that the excess pressure drop cannot be generally expressed by Eq. (*21*), but, instead, another correction factor, the Coette factor C', appears representing the excess pressure drop in the entrance region under the flow. The dependence of the excess losses on the Reynolds number may then be expressed in the simplest form as a sum of both excess losses, *i.e.*

$$(\Delta P)_E / (1/2 \varrho U^2) = C + C'/\text{Re}^*\,. \tag{47}$$

Corresponding numerical analysis of the flow equations have been made to date only for Newtonian liquids[6] and a power-law liquid[5], $0{\cdot}8 < n^*$. Even for a Newtonian liquid though certain discrepancy exists between the experimental and theoretical values[18] of C and C'. It seems therefore premature to embark on a similar analysis of a non-Newtonian case.

Aside from the original analysis for the Eyring model and the generally known results[1,2] for the power-law model we have utilized also the results for the Bingham model published in paper[8]. The application of the boundary layer results to viscoplastic materials encounters certain difficulties, the essence of which rests in the physical concept of the boundary layer. Outside the boundary layer the flow is usually assumed to be a potential one with the corresponding pattern of the pressure, *i.e.* all components of the stress tensor being equal to zero. However, to make the viscoplastic liquid flow the second invariant of the deformation tensor has to exceed τ_0. Under low stresses the viscoplastic liquid behaves like an elastic body and cannot transmit the isotropic pressure in the usual way. Even in cases when the simplifications imposed by the boundary layer theory were adequate, the boundary layer would develope under the action of substantially different pressure profile than that given by the Bernoulli equation for an inviscid fluid. In this respect the problems of the boundary layer theory of viscoplastic liquid have much in common with the theory of viscoelastic boundary layer[10].

The solution of the entrance region for a Bingham liquid in the proposed way must therefore be regarded as an illustration of a pseudosimilarity approach rather than a rigorous method of attacking the problem of the viscoplastic flow in the entrance region.

Before concluding it remains to investigate the extent to what the results obtained by the Schiller's pseudosimilarity approach may serve as a criterion of applicability of the pseudosimilarity approach to more realistic hydrodynamic models of the entrance-flow region. Discussion of this problem will be based on comparison of the Schiller's approach with its more recent modification due to Campbell and Slattery[19]. As it is known the authors have used the method to make a rather realistic prediction of the axial variation of pressure, the length of the entry region (99% criterion) and the Hagenbach's correction factor. Their results are quite comparable with more exact solutions of the boundary layer in a tube.

The only difference between both methods consists in that the hypothetical validity of the Euler equation in the liquid core (*i.e.* Eq. (*4*)), governing the axial variation of $P(z)$ is replaced by an integral balance of the mechanical energy in the following form

$$\tfrac{1}{2}\varrho \frac{\mathrm{d}}{\mathrm{d}z}\int_0^R v_z^3 r \,\mathrm{d}r = -\frac{\mathrm{d}P}{\mathrm{d}z}\frac{UR^2}{2} + \int_0^R \tau_{rz}\left(-\frac{\partial v_z}{\partial r}\right) r \,\mathrm{d}r\,. \qquad (48)$$

This equation enables the pressure drop to be eliminated by combining Eqs (*48*) and (*7*) and the thickness of the boundary layer as well as some other characteristics of the entrance-flow region to be determined by integration of a single ordinary differential equation in ξ and δ^*. This equation contains numerical constants depending on the developed velocity profile and generally on non-linear viscosity function $\vartheta[p]$.

In boths methods compared we had made identical assumptions regarding the velocity profile in the boundary layer, which are equivalent to the assumption of a linear radial profile of the shear stress τ_{rz} in the boundary layer. Since the pseudosimilarity approach compares solutions corresponding to $n^* =$ idem for a developed laminar profile (which ensures also a very close similarity of the velocity profiles in the boundary layer), the individual terms in Eq. (*48*) will therefore have very similar values even under considerably different viscosity functions.

Having thus shown the utility of the pseudosimilarity approach for the Schiller's method one can expect that similar results will be achieved also with the Campbell–Slattery method since their differences do not touch the prerequisites of the pseudosimilarity approach but merely describe more adequately the relation between the variations of the kinetic energy and the pressure changes in the process of formation

TABLE I

The Results of a Numerical Solution for the Eyring Model

A	0·1000	0·7000	1·0000	1·4000	1·8000	2·6000	4·0000	5·0000
B	0·1001	0·7388	1·1154	1·7281	2·5318	5·1169	16·952	40·308
$p(A)$	0·1001	0·7586	1·1752	1·9043	2·9422	6·6947	27·290	74·203
n^*	0·9978	0·9033	0·8233	0·7104	0·6067	0·4476	0·2907	0·2292
γ_1	1·9994	1·9736	1·9476	1·9028	1·8498	1·7345	1·5519	1·4530
α_1	0·3332	0·3279	0·3226	0·3129	0·3011	0·2730	0·2213	0·1891
α_2	0·1666	0·1626	0·1586	0·1515	0·1428	0·1225	0·0868	0·0663
β_1	0·4665	0·4605	0·4543	0·4430	0·4288	0·3938	0·3256	0·2806
β_2	0·1333	0·1306	0·1278	0·1227	0·1164	0·1015	0·0738	0·0570
δ^* for ζ:								
(0·2)	0·00197	0·00310	0·00357	0·00404	0·00436	0·00458	0·00425	0·00383
(0·4)	0·01018	0·01233	0·01328	0·01408	0·01440	0·01395	0·01171	0·01001
(0·6)	0·02893	0·03146	0·03241	0·03279	0·03225	0·02937	0·02274	0·01865
(0·7)	0·04370	0·04603	0·04666	0·04635	0·04487	0·03980	0·02975	0·02396
(0·8)	0·06276	0·06457	0·06461	0·06316	0·06030	0·05222	0·03783	0·02998
(0·9)	0·08652	0·08743	0·08655	0·08348	0·07873	0·06675	0·04700	0·03672
(1·0) ζ_E	0·11517	0·11477	0·11261	0·10739	0·10021	0·08339	0·05727	0·04416
C	1·1549	1·0588	0·9914	0·9024	0·8184	0·67425	0·49207	0·40465

of the velocity profile which are (for a given thickness of the boundary layer) superimposed in both methods in the same manner.

This is evidenced also by direct test of pseudosimilarity of the flow in the entrance region for the Campbell–Slattery method. This method, however, can be modified succesfully so as to involve some additional important aspects of the non-Newtonian entrance-flow region, and, in particular, the dissipation of mechanical energy by the elongation flow of the liquid core. Since these possibilities will be examined in our next paper, we have confined ourselves here to the presented qualitative analysis.

APPENDIX

The Characteristics of the Flow in a Tube for the Eyring Model

The viscosity function of the Eyring model is given by

$$p = \sinh(\vartheta) = \sum \vartheta^{(2i-1)}/(2i-1)! \qquad (D1a,b)$$

The summation sign in this equation (which is absolutely convergent for $\vartheta \leqq 1$) as well as those in the following text indicate summation over all natural numbers.

The individual characteristics of a fully developed flow in a tube introduced in the paper take for the Eyring model the following form as functions of the radial dimensionless coordinate $y = r/R$ and the dimensionless pressure drop $A = \tau_w/\tau_I$.

$$B(A) = (\cosh(A) - 2 - (\sinh(A) - \cosh(A) - 1)/A)/A)/A = \qquad (D2a)$$

$$= \sum \frac{A^{2i-1}}{(2i+2)\,(2i-1)!} \qquad (D2b)$$

$$f(y, A) = S(A)\,(\cosh(A) - \cosh(Ay))/A^2 \qquad (D3a)$$

$$= S(A) \sum A^{2i-1}(1 - y^{2i})/(2i)! \qquad (D3b)$$

where $S(A)$ is an auxiliary function

$$S(A) = \frac{A^2}{\cosh(A) - 1} = 1/\sum(A^{2i-2}/(2i)!)$$

$$\int_0^1 f(y)\, y\, dy = U/U_{max} = S(A)/A \qquad (D4a,b)$$

$$f'(1) = \sinh(A)/B = \frac{\sum A^{2i-2}/(2i-1)!}{\sum A^{2i-2}/[(2i+1)\,(2i-1)!]} \qquad (D5a,b)$$

$$\int_0^1 f(y)\, dy = (S(A)/A^2)\,(\cosh(A) - \sinh(A)/A) = \qquad (D6a)$$

$$= (S(A)/A^2) \sum A^{2i}/((2i+1)\,(2i-1)!) \qquad (D6b)$$

$$\int_0^1 f^2(y)\,\mathrm{d}y = (S^2(A)/A^4)\,(\cosh^2(A) - 3/2\cosh(A)\sinh(A)/A + 1/2) = \tag{D7a}$$

$$= S^2(A)\sum_{j=1}^{\infty} A^{2j-2}\sum_{i=1}^{j}\frac{2i((2i+2+(2j-2i+3)(2j+3))}{(2j+3)(2i+1)!\,(2j-2i+3)!} \tag{D7b}$$

$$\int_1^0 f^2(y)\,y\,\mathrm{d}y =$$

$$= (S^2(A)/A^4)\,(\cosh^2(A)/2 - 2\cosh(A)\,.\,((\sinh(A) - (\cosh(A)/A) +$$

$$+ 1/4 + \sinh(A)\,.\,\cosh(A)/(2A) - (\cosh^2(A) + \sinh^2(A) - 1)/(8A^2)) \tag{D8a}$$

$$= S^2(A)\sum_{j=1}^{\infty} A^{2j-2}\sum_{i=1}^{j}\frac{i(1/(2i+2) - 1/((j+2)(2j-2i+4)))}{(2i)!\,.\,(2j-2i+2)!} \tag{D8b}$$

$$n^*(A) = 1/(\sinh(A)/B(A) - 3) = \tag{D9a}$$

$$= \frac{\sum A^{2i-2}/((2i+2)\,.\,(2i-1)!)}{\sum A^{2i-2}/((2i+2)\,.\,(2i-2)!)}\,. \tag{D9b}$$

The differential equation (*33*) was integrated with the parameters defined by the above equations for a number of values of A and some results are summarized in Table I.

LIST OF SYMBOLS

A	dimensionless pressure drop, Eq. (*27a*), (*1*)
B	dimensionless flow rate, Eq. (*27b*), (*1*)
C	the Hagenbach's correction factor, Eq. (*21*), (*1*)
C'	the Couette correction factor, Eq. (*47*), (*1*)
D	deformation rate-scalar, (s^{-1})
D_I	material constant of the GNF, Eq. (*26*), (s^{-1})
$f(y)$	normalized velocity in the boundary layer, Eq. (*9*), (*1*)
K	consistency coefficient for the power-law model of the GNF, (dyn cm^{-2} s^n)
K^*	dimensional parameter of automorphous models, Eq. (*38*), (dyn cm^{-2} s^n)
L	tube length, (cm)
n	flow index of the power-law model of the GNF, (*1*)
n^*	apparent flow index in automorphous models, Eq. (*37*), (*1*)
P	isotropic pressure, (dyn cm^{-2})
$(\Delta P)_E$	excess pressure drop in the entrance region Eq. (*21*), (dyn cm^{-2})
p	normalized viscosity function of the GNF, Eq. (*26*), (*1*)
Q	volume flow rate in the tube, (cm^3 s^{-1})
r	radial coordinate, (cm)
R	radius of tube, (cm)
Re^*	non-Newtonian flow Reynolds number in a tube, Eq. (*41*), (*1*)
U	average velocity in a tube, (cm s^{-1})
U_{max}	maximum velocity in a tube, (cm s^{-1})
v_z, v_r	velocity components in radial coordinates, (cm s^{-1})

$v_D(r)$ fully developed velocity profile of non-Newtonian laminar flow in a tube, (cm s^{-1})
$v_0(z)$ axial velocity in the potential core of the entrance region, (cm s^{-1})
y normalized radial coordinate in the boundary layer, Eq. (*10*), (*1*)
z radial coordinate, (cm)
z_E length of the entrance region, (cm)
$\alpha_{1,2}$ constants of the Schiller's model of the entrance region, Eqs (*15*), (*16*), (*1*)
$\beta_{1,2}$ constants of the Schiller's-model of the entrance region, Eqs (*17*), (*18*), (*1*)
γ_1 constants of the Schiller's-model of the entrance region, Eq. (*30*), (*1*)
δ local boundary layer thickness, (cm)
δ^* normalized boundary layer thickness, Eq. (*14*), (*1*)
$\vartheta[p]$ normalized viscosity function of the GNF, Eq. (*26*), (*1*)
ϱ density, (g cm^{-3})
$\tau[D]$ viscosity function of the GNF, (dyn cm^{-2})
τ_{zr} shear stress in the tube, (dyn cm^{-2})
τ_I material constant of the GNF, Eq. (*26*), (dyn cm^{-2})
τ_0 yield stress material constant of a viscoplastic liquid, (dyn cm^{-2})
τ_w shear stress on the tube wall, (dyn cm^{-2})
$\mathfrak{P}[\mathfrak{F}]$ shear stress on the tube wall under laminar flow without wall effects, consistency variable, Eq. (*25*), (dyn cm^{-2})
$\mathfrak{F}$ consistency variable. Eq. (*25*), (s^{-1})

Subscript "a" indicates a functional dependence for an automorphous (power-law) model. The square bracketts indicate the material-and-related functions.

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